

# Low-dimensional dynamics embedded in a plane Poiseuille flow turbulence Traveling-wave solution is a saddle point?

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The instability of a streak and its nonlinear evolution are investigated by direct numerical simulation (DNS) for plane Poiseuille flow at  $Re = 3000$ . It is suggested that there exists a traveling-wave solution (TWS). The TWS is localized around one of the two walls and notably resemble to the coherent structures observed in experiments and DNS so far. The phase space structure around this TWS is similar to a saddle point. Since the stable manifold of this TWS is extended close to the quasi two dimensional (Q2D) energy axis, the approaching process toward the TWS along the stable manifold is approximately described as the instability of the streak (Q2D flow) and the succeeding nonlinear evolution. Bursting corresponds to the escape from the TWS along the unstable manifold. These manifolds constitute part of basin boundary of the turbulent state.

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In wall-turbulent shear flows, turbulence production is believed to occur or at least to be triggered in the near-wall region in a deterministic and intermittent way, and also to be related to coherent structures: bursting [1]. Although the mechanism of the bursting has not been well understood, the coherence of this process suggests that a low-dimensional dynamics is embedded even in fully-developed turbulence [2]. This low-dimensionality observed in subcritical flows [3] is quite different from weakly nonlinear stage of supercritical flows such as Bénard convection [4], because the fully-developed turbulence coexists. That is, the low-dimensional dynamics should be connected to the huge-dimensional dynamics globally.

This global structure seems to be hopelessly complicated. However, a minimal flow unit for maintaining turbulence was found by Jinémez and Moin [5] in plane Poiseuille flow. Owing to this work we can focus on the elementary process of turbulence in the near-wall region from the perspective of dynamical systems without being bothered by spatial coexistence of various stages. Hamilton et al. [6] examined the minimal flow unit of plane Couette flow and found that turbulence is sustained by not random but quasi-cyclic process passing quiescent and activated periods in turn. The quiescent period is relatively laminar where coherent structures, the so-called streaks, are formed, develop and become unstable. Then their instability triggers an activated or turbulent period. They called this process self-sustaining process (SSP). Note that the Reynolds numbers ( $Re$ ) used in their work are quite low, so they observed spatial chaos rather than fully-developed turbulence.

This picture seems to work well even in realistic situations. There are, however, still open questions. One is on the nonlinear evolution after the instability of streaks and another is why the unstable streaks are regenerated. The low-dimensional model introduced heuristically by

Waleffe [7] has partially answered to these. The keys are the existences of steady solutions and a homoclinic orbit. This model, however, is wholly low-dimensional and not related to fully-developed turbulence.

Recently, Waleffe obtained traveling-wave solutions (TWS) in plan Poiseuille flow by continuing from steady solutions in plane Couette flow for stress boundary conditions [8]. These solutions are quite similar in shape to the coherent structures educed by Joeng et al. [9] in direct numerical simulation (DNS), although the boundary conditions are different.

DNS and experiments have revealed the existence of coherent structures in the near-wall region that are intimately linked to the maintenance of turbulence. In fact the sinuous evolution of a streak observed in DNS appears to trigger off the formation of longitudinal coherent structures and the succeeding bursting. This process is also the main part of the SSP picture. Schoppa and Hussain [10] showed that the instability of ejected low speed streaks directly generates new streamwise vortices, internal shear layers and arch vortices by examining stability of a vortex-less streak [11]. The resulting 3D vortex geometry is identical to that of the dominant coherent structures educed from the near-wall turbulence, too. They concluded that vortex-less streaks are the main breeding ground for new streamwise vortices, commonly accepted as dominant in turbulence production.

In this paper, we try to elucidate the relation of the conceptual ingredients of the SSP picture such as the instability of streaks and coherent structures with TWS, homoclinic nature, etc. in a more realistic situation by means of DNS.

The numerical scheme adopted here is basically the same as those of Kim et al. [12]. For all simulations reported, we use periodic boundary conditions in  $x$  and  $z$  with period lengths  $L_x = \pi \sim 420^+$  and  $L_z = 0.4\pi \sim 170^+$ , and apply no-slip boundary conditions at  $y = \pm 1$ :

$x$ ,  $y$  and  $z$  represent the streamwise, normal to the walls and spanwise directions, respectively. The value of Reynolds number is fixed to 3000. Though somewhat large, this system belongs to the minimal flow unit [5]. The flow is maintained by a constant volume flux. As aliasing errors are removed by 1/2 phase-shift method, the effective modes are  $30 \times 30$  Fourier modes in  $x$  and  $z$ , and 65 Chebychev polynomials in  $y$ .

In Fig. 1, the evolution of Reynolds stress integrated over the horizontal plane,  $|\langle u_x u_y \rangle(y, t)|$  is shown. Turbulent fluctuations are produced close to the walls and ejected into the center region. These sudden production and ejection of fluctuations are typical bursting processes [13]. Bursting on both walls seems to occur independently. In the lower half region, an especially quiescent period where the flow is laminarized and a prominent streak develops, lasts about ten times as long as ordinary one. Even in this period, bursting continues to occur in the upper region. This quasi-cyclic and intermittent occurrence of bursting is reminiscent of the chaotic homoclinic orbit [14].

To describe the low-dimensional dynamics suggested, we introduce the 2D phase space spanned by the quasi-2D (Q2D) and 3D components of kinetic energy of the normal velocity,  $u_y$ , that is,  $E_y^{\text{Q2D}} = \int_V (u_y^{\text{Q2D}})^2 dV/2$  and  $E_y^{\text{3D}} = \int_V (u_y^{\text{3D}})^2 dV/2$ , where  $V$  denotes the whole volume of the computation. Here we decompose the velocity field into two parts: Q2D flow  $\mathbf{u}^{\text{Q2D}}(y, z, t)$ , and 3D flow  $\mathbf{u}^{\text{3D}}(x, y, z, t)$ . The latter is constructed by Fourier modes only with non-zero streamwise wave number. The velocity field restricted in Q2D space, is damped to the laminar plane Poiseuille flow monotonically for all  $Re$  [15]. In this sense, there is no fixed point on the Q2D axis except for the origin.

Hamilton et al. [6] defined the streak and the longitudinal vortex as the  $x$  component( $u_x^{\text{Q2D}}$ ) of Q2D flow and its  $y, z$  components ( $u_y^{\text{Q2D}}, u_z^{\text{Q2D}}$ ), respectively for the minimal Couette turbulence. They also regarded 3D flow as turbulent components. The simplicity of these definitions is a little bit curious because coherent structures are observed in the near-wall region and also three-dimensional. These definitions, however, conceptually work well. This suggests that quiescent stages or generating processes of coherent structures are well described in a low dimensional phase space.

To see the global behavior of the system in phase space, we examine the nonlinear evolutions of small 3D disturbances superposed to an artificially obtained Q2D velocity field that seems to be a prototype of a well-developed streak, as done by Schoppa and Hussain [10]. They used an analytic streak solution without longitudinal vortex [11]. In contrast, our streak solution is constructed by removing the  $x$ -dependent modes, i.e. the 3D component, from the streak-dominated velocity field obtained in DNS at  $t = 337$  in Fig. 1 in order to compare the obtained results with the real situation.

The initial condition is as follows:

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^{\text{Q2D}}(y, z, 0) + \sqrt{F_{ac}} \frac{\|u_y^{\text{Q2D}}(0)\|}{\|u_y^{\text{3D}}(0)\|} \mathbf{u}^{\text{3D}}(\mathbf{x}, 0), \quad (0.1)$$

where  $\|f\| \equiv (\int_V f^2 dV)^{1/2}$  and  $\mathbf{u}^{\text{3D}}(\mathbf{x}, 0)$  is a solenoidal random vector field with a given broad spectrum and random phase. The following results are qualitatively independent of the form of  $\mathbf{u}^{\text{3D}}(\mathbf{x}, 0)$ .

We examine the dependence of the evolutions on the the relative amplitude,  $F_{ac}$  of the 3D, i.e.,  $x$ -dependent disturbance. In Fig. 2, the evolutions of  $E_y^{\text{Q2D}}(t)$  and  $E_y^{\text{3D}}(t)$  for several values of  $F_{ac}$  are shown. After initial transient stage where the most growing mode is selected, 3D disturbance ( $E_y^{\text{3D}}(t)$ ) seems to grow exponentially and then saturate to an equilibrium state in damped-oscillating. Finally the disturbance chooses either the route to laminar plane Poiseuille flow or that to turbulent state depending on whether the value of  $F_{ac}$  is smaller or larger than the critical value. We refer to this critical value, which is close to  $F_{ac}^I = 9.116010224 \times 10^{-8}$ , as  $F_{ac}^c$ . The passing time till the final abrupt damping or growth gets longer as  $F_{ac}$  approaches  $F_{ac}^c$ . The exponential growth in the initial stage corresponds to the instability of the streak, while the streak ( $E_y^{\text{Q2D}}(t)$ ) continues to be damped.

Figure 3 shows the evolutions in the phase space spanned by  $E_y^{\text{Q2D}}$  and  $E_y^{\text{3D}}$  for several values of  $F_{ac}$ . It is easy to see that a fixed point like a saddle-focus exists. The trajectory is oscillating but approaches monotonically the expected fixed point.

The difference between two solutions for different  $F_{ac}$  increases exponentially with the roughly constant growth rate,  $\sigma_+ = 0.037$  even in the approaching period. This suggests that the evolution of  $E_y^{\text{3D}}$  is described as a motion around a saddle point like  $(dX/dt, d(Z_r + iZ_i)/dt) = (\sigma_+ X, (-\sigma_- + i\omega)(Z_r + iZ_i))$ , where  $i$  is imaginary unit and  $\sigma_+, \sigma_-$  are positive constants.

The complex damping rate could be also estimated, but the existence of another monotonically-damping mode makes an estimate difficult (see Fig. 2). A rough estimate shows that the ratio of these two damping rates is about 2: the damping rates are -0.004 for the real mode and -0.008 for the complex one. Therefore the dimension of the contracted space around the fixed point is at least larger than 4. This dynamics is more complicated than Waleffe's model.

Both the stable and unstable manifolds of the saddle point constitute a separatrix that separates turbulent and laminar states. The stable manifold is extended close to the Q2D axis. Because of this closeness, the stability of streaks, i.e., of a Q2D solution, is well recognized even in fully-developed wall turbulence like in SSP.

We infer that for  $F_{ac}^c$  the trajectory reaches the fixed point. In physical space, the solution corresponding to this point has a 3D shape and moves steadily with the velocity  $v = 0.75 \pm 0.05$  in the streamwise direction. Thus the fixed point must be TWS. In this sense, this solution

is not a fixed-point but a periodic orbit. Since we cannot yield the TWS by DNS in a strict sense anyway, we regard the saturated state obtained for  $F_{ac}^I$  as TWS hereafter.

This TWS is notably resemble in shape to those obtained by Waleffe for stress boundary conditions as can be seen in Fig. 4, although our  $Re$  is about ten times as large as Waleffe's. The streamwise wave length of Waleffe's solution is almost the same as ours and the spanwise periodicity is 1.67 times as long as ours. Note that while Waleffe's are symmetric with respect to the centerline of the channel, our TWS is confined to the lower wall (see Fig. 5). Our TWS is also so tall in height that it reaches to the upper boundary of the log-law region and is not localized to the near-wall region. Furthermore, in wall turbulence an interval between adjacent streaks is about 100 in wall unit on average. If TWS is linked to the coherent structures, there could exist TWSs with shorter height and/or spanwise periodicity than that of the TWS found by us. (See Fig. 6(c). Two ejections are observed on the upper wall.) This suggests that other TWSs confined to the near-wall region exist and these TWSs should be common in realistic turbulence.

To compare the quiescent period of the turbulence and the low dimensional dynamics around the TWS, we also project the evolution of the turbulence onto the 2D phase space as shown in Fig.3. Here we only use the energies contained in the lower half volume because the other turbulent evolution occurs independently in the upper half volume (see Fig. 6(c)). It is quite surprising that the projected trajectory is close to the unstable manifold, because wall turbulence is usually coherent or in order only in the near-wall region. In the bursting period the system goes away from the TWS along the unstable manifold. This suggests that the instability not of a streak but of a TWS is the origin of the bursting process. In Fig. 6 we show the solutions on the unstable manifold and the trajectory of turbulence marked in Fig. 3. From these figures, the resemblance is apparent.

The most unstable mode of the streak has the same wavelength as the TWS itself for  $L_x = \pi$ . This seems to imply that the instability of streaks directly breed coherent structures, i.e., TWS. This is, however, not always true. Indeed, when  $L_x$  is doubled, the most unstable mode to the streak is saturated to another TWS with double the periodicity: the streamwise wave numbers of the most unstable modes for  $L_x = \pi$  and  $2\pi$  are the same. Moreover, we believe that bursting is the escaping process from TWS along the unstable manifold. Thus the stability of TWS is more important than that of streak. The further study of the former instability should elucidate bursting, i.e., turbulence generation and also the determination of streak intervals in wall turbulence.

As mentioned above, there may exist many TWSs. These TWSs may be connected with each others through turbulent or activated periods, although we have not understood how they are. This generalized homoclinic or "multi-clinic" nature of wall turbulence seems to be the substance of coherent structures and quasi-cyclic evolu-

tion, SSP.

In the context of control of turbulence generation, the existence of separatrix supports the laminarization of turbulent flow by forcing the flow to be two-dimensional at least in the near-wall region *e.g.* by means of ripples or suction [13,16].

The computations have been performed on the NEC/SX4 of YITP, Kyoto Univ.

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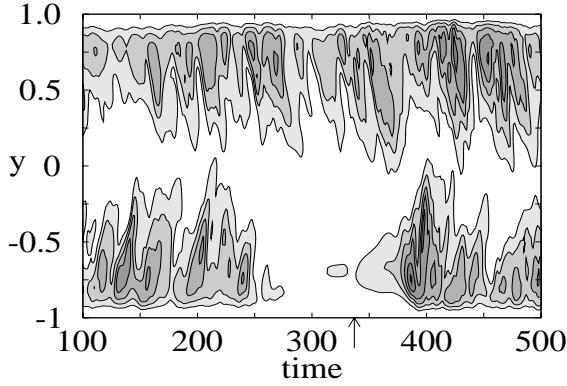


FIG. 1. Evolution of Reynolds stress integrated over horizontal plane. Shade indicates  $|\langle u_x u_y \rangle(y, t)| \geq 0.5$ .

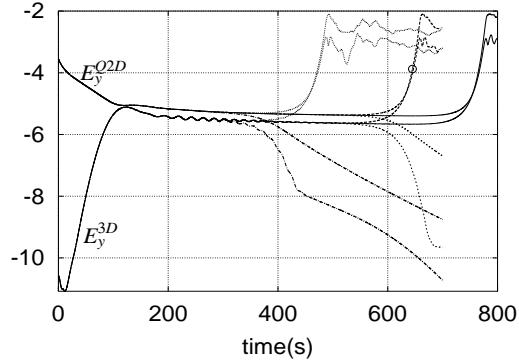


FIG. 2. Linear-log plot of evolutions of  $E_y^{Q2D}(t)$  and  $E_y^{3D}(t)$  for  $F_{ac}^I = 9.116010224 \times 10^{-8}$  (solid line),  $F_{ac}^{II} = 9.1160104 \times 10^{-8}$  (long-dashed line;  $\circ$  indicates  $t = 645$ ),  $F_{ac}^{III} = 9.11601 \times 10^{-8}$  (short-dashed line),  $F_{ac}^{IV} = 9.1162 \times 10^{-8}$  (dotted line),  $F_{ac}^V = 9.114 \times 10^{-8}$  (dash-dotted line).

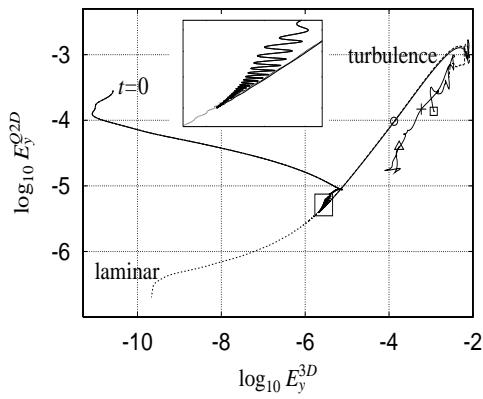


FIG. 3. Log-log plots of evolutions of  $(E_y^{Q2D}(t), E_y^{3D}(t))$  in 2D phase space. Correspondence between line and  $F_{ac}$  is the same as that in Fig.2. Thick solid line shows the evolution of the turbulence from  $t = 300(\Delta)$  to  $t = 450(\square)$  via  $t = 380(+)$  in Fig.1.  $\circ$  indicates  $t = 645$  for  $F_{ac}^{II}$ .

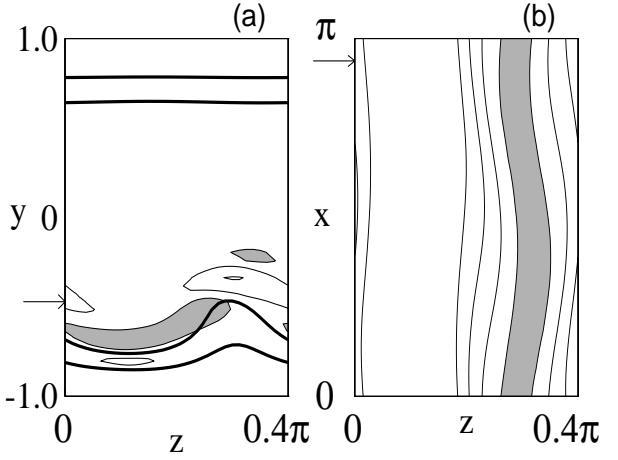


FIG. 4. Snapshot of TWS at  $t = 600$  for  $F_{ac}^I$ . (a) longitudinal vorticity  $\omega_x(y, z)$  at  $z^+ = 398$ . (b)  $u_x(x, z)$  at  $y^+ = 71$ . Shade indicates  $u_x < 0.65$  or  $\omega_x < -0.05$ . Arrows indicate the positions of the section. Thick solid lines are for  $u_x = 0.4$  and  $0.6$ .

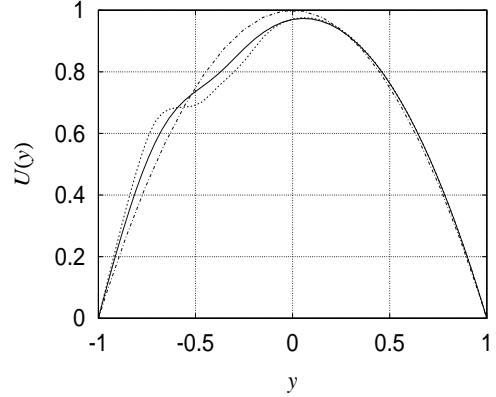


FIG. 5. Mean streamwise velocity profile  $U(y, t)$ . Solid line: TWS at  $t = 600$  for  $F_{ac}^I$  in Fig.2. Dash-dotted line: laminar plane Poiseuille flow. Dashed line: a snapshot close to the unstable manifold at  $t = 650$  for  $F_{ac}^{II}$ .

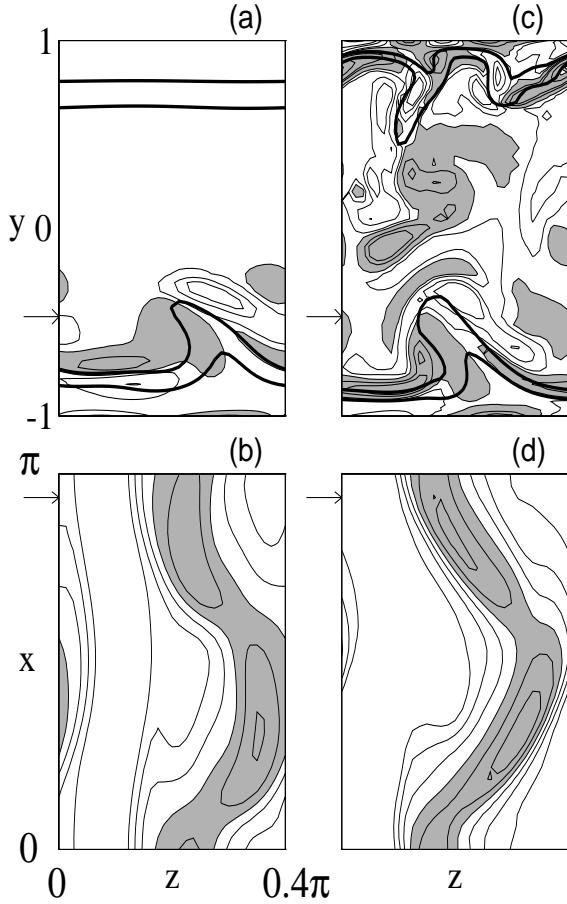


FIG. 6. Snapshots. (a),(b): close to the unstable manifold (○ in Fig.3), (c),(d): in the bursting process of turbulence(+) in Fig.3. (a),(c)  $\omega_x(x, y)$  at  $z^+ = 398$ . (b),(d)  $u_x(x, z)$  at  $y^+ = 71$ . Shade indicates  $u_x < 0.65$  or  $\omega_x < -0.1$ . Thick solid lines are for  $u_x = 0.4$  and  $0.6$ . The phase of (c) and (d) is shifted for ease of comparison.